GALOIS THEORY FINAL EXAM

November 8 2024

This exam is of **50 marks** and is **3 hours long**. Please **read all the questions carefully**. Please feel free to use whatever theorems you have learned in class after stating them clearly.

1. Using the Lindemann-Weierstrass theorem as proved in class show that if $\alpha_1, \ldots, \alpha_n$ are linearly independent algebraic numbers over $\mathbb{Q}$, then $e^{\alpha_1}, \ldots, e^{\alpha_n}$ are algebraically independent, that is, there is no f in $\mathbb{Q}[x_1, \ldots, x_n]$ such that $f(e^{\alpha_1}, \ldots, e^{\alpha_n}) = 0$. (5)

2. Is $\sin(3/2)$ algebraic or transcendental? Justify your answer. (5)

3a. Compute the Galois group of $X^4 - 2$. (5)
3b. Determine all intermediate extensions. (5)
3c. If α is a root of $X^4 - 2$ is it constructible? (5)

4. Let K/F be an infinite Galois extension with $G = Gal(K/F)$. Show that if H is a subgroup then $\bar{H}$, the closure of H in the Krull topology is given by

$$\bar{H} = \bigcap_{N \in \mathcal{N}} HN$$

where $\mathcal{N}$ is the set of subgroups of G such that $N = Gal(K/E)$ with E/F finite Galois. (5)

5. Suppose F is a field containing the n^{th} roots of 1. Let $a \in F$. Show that $X^n - a$ is irreducible if and only if a is not an m^{th} power for any $m > 1$ such that $m|n$. (5)

6. Show that if K is the splitting field of a separable polynomial f then $Gal(K/F)$ is a *transitive* subgroup of S_n - that is, if $i, j \in \{1, \ldots, n\}$ then there exists $\sigma \in Gal(K/F)$ such that $\sigma(\alpha_i) = \alpha_j$, where $\alpha_1, \ldots, \alpha_n$ are the roots of f. (5)

7. Consider the polynomial

$$f(x) = x^3 + 5x + 1$$

- Is it irreducible? Justify your answer. (3)
- What is its discriminant? (3)
- What is the Galois group of the splitting field of f? (4)